SYMMETRIES OF THE CUBIC AND METHODS OF TREATING THE IRREDUCIBLE CASE.

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The study of the internal structure of quantics is always interesting, and in a higher sense practically useful, especially if we succeed in exhibiting, by a suitable notation, the symmetries which must exist in them. The following study of the cubic has been undertaken with this view, in part, hoping thereby to discover a method of treating the irreducible case, if one root at least is rational. The ordinary definition of the irreducible case is, that the discriminant is negative. The cube roots in Cardan's formula will then be imaginary, although the three roots of the cubic are real, and may be found by the trisection of an angle. I agree, however, with Guido Weichold, who says, in his elaborate article on the irreducible case (American Journal of Mathematics, Vol. I, p. 32), that it is an essential condition for it, that one root be rational, and that some method of approximation, among which he includes the trigonometrical by trisection of an angle, must be used in case of irrational roots. The oldest method of treating the irreducible case is that of Bombelli, which is as follows:—

Let x_1 be one root of the cubic

$$0 = x^3 - px + q;$$

then, by Cardan's formula,

$$x_1 = \sqrt[3]{-\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}q^2 - \tfrac{1}{27}p^3}} + \sqrt[3]{-\tfrac{1}{2}q - \sqrt{\tfrac{1}{4}q^2 - \tfrac{1}{27}p^3}},$$

which according to Bombelli is

$$= \tfrac{1}{2}x_1 + \sqrt{\tfrac{1}{4}x_1^2 - \tfrac{1}{3}p} + \tfrac{1}{2}x_1 - \sqrt{\tfrac{1}{4}x_1^2 - \tfrac{1}{3}p};$$

and, if $\qquad 1^{\frac{1}{3}} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}, \quad 1^{\frac{2}{3}} = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3},$

the other two roots become

$$x_2 = \left(\tfrac{1}{2}x_1 + \sqrt{\tfrac{1}{4}x_1^2 - \tfrac{1}{3}p}\right)1^{\frac{1}{3}} + \left(\tfrac{1}{2}x_1 - \sqrt{\tfrac{1}{4}x_1^2 - \tfrac{1}{3}p}\right)1^{\frac{2}{3}} = -\tfrac{1}{2}x_1 + \tfrac{1}{2}\sqrt{4p - 3x_1^2},$$

$$x_3 = \left(\tfrac{1}{2}x_1 + \sqrt{\tfrac{1}{4}x_1^2 - \tfrac{1}{3}p}\right)1^{\frac{2}{3}} + \left(\tfrac{1}{2}x_1 - \sqrt{\tfrac{1}{4}x_1^2 + \tfrac{1}{3}p}\right)1^{\frac{1}{3}} = -\tfrac{1}{2}x_1 - \tfrac{1}{2}\sqrt{4p - 3x_1^2}.$$

Although this method assumes the previous knowledge of one root, it is as satisfactory and convenient as any method yet proposed. This will be better understood after we have studied the internal relations of the general cubic

$$\begin{aligned} 0 = (x, y)_3 &= ax^3 + bx^2y + cxy^2 + dy^3 \\ &= (xy_1 - x_1y)(xy_2 - x_2y)(xy_3 - x_3y), \end{aligned} \tag{1}$$

in which we suppose the coefficients a, b, c, d to be integers, so that one root at least, for instance $x_1 y_1^{-1}$, may be rational. We have, by Taylor's theorem,

$$(x + \Delta x, y)_3 = (x, y)_3 + \frac{\partial}{\partial x}(x, y)_3 \,\Delta x + \tfrac{1}{2}\frac{\partial^2}{\partial x^2}(x, y)_3 \,\Delta x^2 + \tfrac{1}{6}\frac{\partial^3}{\partial x^3}(x, y)_3 \,\Delta x^3$$

$$= ax^3 + bx^2y + cxy^2 + dy^3$$

$$+ (3\,ax^2 + 2\,bxy + cy^2)\,\Delta x + (3\,ax + by)\,\Delta x^2 + a\,\Delta x^3. \qquad (2)$$

If in this we put $x = -b$, $y = 3a$ and $\Delta x = 3a\,xy^{-1} + b$, it becomes

$$0 = (3a\,xy^{-1}, 3a)_3 = (3ay^{-1})^3 (x, y)_3$$

$$= a(3a\,xy^{-1} + b)^3 + \frac{\partial}{\partial x}(-b, 3a)_3 (3a\,xy^{-1} + b) + (-b, 3a)_3$$

$$= a(3a\,xy^{-1} + b)^3 - 3a(b^2 - 3ac)(3a\,xy^{-1} + b) + a(2b^3 - 9\,abc + 27\,a^2d)$$

$$= (3a\,xy^{-1} + b)^3 - 3(b^2 - 3ac)(3a\,xy^{-1} + b) + 2b^3 - 9\,abc + 27\,a^2d. \qquad (3)$$

If we vary y instead of x, we have

$$0 = (x, y + \Delta y)_3 = (x, y)_3 + \frac{\partial}{\partial y}(x, y)_3 \,\Delta y + \tfrac{1}{2}\frac{\partial^2}{\partial y^2}(x, y)_3 \,\Delta y^2 + \tfrac{1}{6}\frac{\partial^3}{\partial y^3}(x, y)_3 \,\Delta y^3$$

$$= ax^3 + bx^2y + cxy^2 + dy^3$$

$$+ (bx^2 + 2\,cxy + 3\,dy^2)\,\Delta y + (cx + 3\,dy)\,\Delta y^2 + d\,\Delta y^3; \qquad (4)$$

which, by placing $x = 3d$, $y = -c$, $\Delta y = c + 3d\,x^{-1}y$, becomes

$$0 = (3d, 3d\,x^{-1}y)_3 = (3dx^{-1})^3 (x, y)_3$$

$$= d(c + 3d\,x^{-1}y)^3 + \frac{\partial}{\partial y}(3d, -c)_3 (c + 3d\,x^{-1}y) + (3d, -c)_3$$

$$= (c + 3d\,x^{-1}y)^3 - 3(c^2 - 3bd)(c + 3d\,x^{-1}y) + 2c^3 - 9\,bcd + 27\,ad^2. \qquad (5)$$

For greater conciseness, let us write

the cubic variant, $\quad 2b^3 - 9abc + 27a^2d = A$; (6)

" quadratic variant, $\quad b^2 - 3ac = A'$; (7)

" " retrovariant, $\quad c^2 - 3bd = D'$; (8)

" cubic " $\quad 2c^3 - 9bcd + 27ad^2 = D.$ (9)

Then we have

$$0 = (3a\,xy^{-1} + b)^3 - 3A'(3a\,xy^{-1} + b) + A, \qquad (3')$$

$$0 = (c + 3d\,x^{-1}y)^3 - 3D'(c + 3d\,x^{-1}y) + D. \qquad (5')$$

Since by (1)

$$a = y_1 y_2 y_3, \qquad b = -x_1 y_2 y_3 - y_1 x_2 y_3 - y_1 y_2 x_3,$$

$$c = x_1 x_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3, \quad d = -x_1 x_2 x_3 ; \tag{10}$$

the unknown quantities $3a\, xy^{-1} + b$, $c + 3d\, x^{-1}y$ will have at least one integer value, if x_1 and y_1 are integers.

Assuming now, in (3′)

$$0 = 3a\, xy^{-1} + b + b' + b'', \tag{11}$$

and in (5′)

$$0 = c'' + c' + c + 3d\, x^{-1}y, \tag{12}$$

where b', b'' and c', c'' are real or imaginary conjugates, whose sums are integers, then (3′) and (5′) become

$$0 = (b' + b'')^3 - 3A' (b' + b'') - A, \tag{3''}$$

$$0 = (c' + c'')^3 - 3D' (c' + c'') - D. \tag{5''}$$

We have, then,

$$A = b'^3 + b''^3, \tag{6'}$$

$$A' = b' b'', \tag{7'}$$

$$D' = c' c'', \tag{8'}$$

$$D = c'^3 + c''^3 ; \tag{9'}$$

and consequently

$$\left.\begin{matrix} b' \\ b'' \end{matrix}\right\} = \sqrt[3]{\tfrac{1}{2} A \pm \sqrt{\tfrac{1}{4} A^2 - A'^3}} = \sqrt[3]{\tfrac{1}{2} A \pm \tfrac{3}{2} a \sqrt{\varDelta_3}}, \tag{13}$$

$$\left.\begin{matrix} c' \\ c'' \end{matrix}\right\} = \sqrt[3]{\tfrac{1}{2} D \pm \sqrt{\tfrac{1}{4} D^2 - D'^3}} = \sqrt[3]{\tfrac{1}{2} D \pm \tfrac{3}{2} d \sqrt{\varDelta_3}}, \tag{14}$$

where $\varDelta_3$ denotes the discriminant. If this is negative, for the determination of either b', b'' or c', c'' a cube root of a complex quantity has to be extracted, and in this consists the whole difficulty of the irreducible case; for, having done this, the roots of the cubic are immediately given by (11) or (12).

Let us now study the mutual relations of the pair b', b'' to the pair c', c''. If (11) gives one root $x_1 y_1^{-1}$, which we suppose rational; then all three roots will be given by the system

$$\begin{aligned} 0 &= 3a\, x_1 y_1^{-1} + b + b' + b'', \\ 0 &= 3a\, x_2 y_2^{-1} + b + b'\, 1^{\frac{1}{3}} + b''\, 1^{\frac{2}{3}}, \\ 0 &= 3a\, x_3 y_3^{-1} + b + b'\, 1^{\frac{2}{3}} + b''\, 1^{\frac{1}{3}}. \end{aligned} \tag{15}$$

Similarly from (12) follow the relations

$$\begin{aligned} 0 &= c'' + c' + c + 3d\, x_1^{-1}y_1, \\ 0 &= c''\, 1^{-\frac{2}{3}} + c'\, 1^{-\frac{1}{3}} + c + 3d\, x_2^{-1}y_2, \\ 0 &= c''\, 1^{-\frac{1}{3}} + c'\, 1^{-\frac{2}{3}} + c + 3d\, x_3^{-1}y_3. \end{aligned} \tag{16}$$

It will be noticed that the cube root of unity factors of b', c' and also of b'', c'' for the same root of the cubic correspond in such a manner that their product is unity. This is in a measure arbitrary, but is adopted because it produces the most perfect symmetry. From (15) we deduce

$$\begin{aligned} 0 &= a^{-1}b + x_1y_1^{-1} + x_2y_2^{-1} + x_3y_3^{-1}, \\ 0 &= a^{-1}b' + x_1y_1^{-1} + x_2y_2^{-1}\, 1^{-\frac{1}{3}} + x_3y_3^{-1}\, 1^{-\frac{2}{3}}, \\ 0 &= a^{-1}b'' + x_1y_1^{-1} + x_2y_2^{-1}\, 1^{-\frac{2}{3}} + x_3y_3^{-1}\, 1^{-\frac{1}{3}}. \end{aligned} \tag{17}$$

Similarly from (16):

$$\begin{aligned} 0 &= y_3x_3^{-1} + y_2x_2^{-1} + y_1x_1^{-1} + cd^{-1}, \\ 0 &= y_3x_3^{-1}\, 1^{\frac{2}{3}} + y_2x_2^{-1}\, 1^{\frac{1}{3}} + y_1x_1^{-1} + c'd^{-1}, \\ 0 &= y_3x_3^{-1}\, 1^{\frac{1}{3}} + y_2x_2^{-1}\, 1^{\frac{2}{3}} + y_1x_1^{-1} + c''d^{-1}. \end{aligned} \tag{18}$$

Then remembering $0 = 1 + 1^{\frac{1}{3}} + 1^{\frac{2}{3}} = 1 + 1^{-\frac{1}{3}} + 1^{-\frac{2}{3}}$, we easily deduce the following:

$$\begin{aligned} b'b'' &= b^2 - 3ac = A', \\ b''b &= b'^2 - 3ac', \\ bb' &= b''^2 - 3ac'', \end{aligned} \tag{19}$$

$$\begin{aligned} bc + b'c' + b''c'' &= 9\, ad, \\ bc' + b'c'' + b''c &= 0, \\ bc'' + b'c + b''c' &= 0, \end{aligned} \tag{20}$$

$$\begin{aligned} c'c &= c''^2 - 3\, b''d, \\ cc'' &= c'^2 - 3\, b'd, \\ c''c' &= c^2 - 3\, bd = D', \end{aligned} \tag{21}$$

Since, therefore,

$$b'c' + b''c'' = 9\, ad - bc,$$

and

$$b'c' \,.\, b''c'' = (b^2 - 3\, ac)\,(c^2 - 3\, bd) = A'D'\,;$$

the products $b'c' = p'$ and $b''c'' = p''$ will be the roots of the quadratic

$$0 = p^2 - (9\,ad - bc)\,p + A'D'. \tag{22}$$

This remarkable resolvent, which bears such perfect symmetry to the cubic that it is unaffected by exchanging a for d, b for c, x for y, and A' for D', has as far as I am aware never been given. Solving it we have

$$\left.\begin{matrix} b'c' \\ b''c'' \end{matrix}\right\} = \left.\begin{matrix} p' \\ p'' \end{matrix}\right\} = \tfrac{1}{2}(9\,ad - bc) \pm \tfrac{1}{2}\sqrt{(9\,ad - bc)^2 - 4\,A'D'}$$
$$= \tfrac{1}{2}(9\,ad - bc) \pm \tfrac{1}{2}\sqrt{\Delta_3}. \tag{23}$$

From (19) and (21) we deduce

$$bA' = bb'b'' = b^3 - 3\,abc = b'^3 - 3\,ab'c' = b''^3 - 3\,ab''c'', \tag{24}$$

$$cD' = cc'c'' = c^3 - 3\,bcd = c'^3 - 3\,b'c'd = c''^3 - 3\,b''c''d. \tag{25}$$

Therefore

$$b' = \sqrt[3]{bA' + 3\,ap'} = \sqrt[3]{b^3 - 3\,a\,(bc - b'c')},$$

$$b'' = \sqrt[3]{bA' + 3\,ap''} = \sqrt[3]{b^3 - 3\,a\,(bc - b''c'')}, \tag{26}$$

and

$$c' = \sqrt[3]{cD' + 3\,p'd} = \sqrt[3]{c^3 - 3\,d\,(bc - b'c')},$$

$$c'' = \sqrt[3]{cD' + 3\,p''d} = \sqrt[3]{c^3 - 3\,d\,(bc - b''c'')}. \tag{27}$$

The coefficients a, b, c, d of the cubic, expressed in terms of the auxiliaries b', b'', c', c'', are as follows:

$$a = \frac{b'^3 - b''^3}{3\,(b'c' - b''c'')} = \frac{1}{3\sqrt{\Delta_3}}\,(b'^3 - b''^3), \tag{28}$$

$$b = -\frac{b'^2c'' - b''^2c'}{b'c' - b''c''} = -\frac{1}{\sqrt{\Delta_3}}\,(b'^2c'' - b''^2c'), \tag{29}$$

$$c = \frac{b'c''^2 - b''c'^2}{b'c' - b''c''} = \frac{1}{\sqrt{\Delta_3}}\,(b'c''^2 - b''c'^2), \tag{30}$$

$$d = -\frac{c''^3 - c'^3}{3\,(b'c' - b''c'')} = -\frac{1}{3\sqrt{\Delta_3}}\,(c''^3 - c'^3). \tag{31}$$

The cubic takes then the form, which is termed the canonical,

$$0 = (x, y)_3 = \frac{1}{3\sqrt{\Delta_3}}\left[(b'x - c''y)^3 - (b''x - c'y)^3\right]; \tag{32}$$

and the three roots of this are given by

$$0 = b'x_1 - c''y_1 - (b''x_1 - c'y_1); \qquad \therefore\ x_1 = c'' - c',\ \ y_1 = b' - b'': \tag{33_1}$$

$$0 = (b'x_2 - c''y_2)\, 1^{\frac{1}{3}} - (b''x_2 - c'y_2)\, 1^{\frac{2}{3}};\ \therefore\ x_2 = c''\, 1^{\frac{1}{3}} - c'\, 1^{\frac{2}{3}},\ \ y_2 = b'\, 1^{\frac{1}{3}} - b''\, 1^{\frac{2}{3}}: \tag{33_2}$$

$$0 = (b'x_3 - c''y_3)\, 1^{\frac{2}{3}} - (b''x_3 - c'y_3)\, 1^{\frac{1}{3}};\ \therefore\ x_3 = c''\, 1^{\frac{2}{3}} - c'\, 1^{\frac{1}{3}},\ \ y_3 = b'\, 1^{\frac{2}{3}} - b''\, 1^{\frac{1}{3}}. \tag{33_3}$$

Substituting $x = c''$ and $y = b'$, and again $x = c'$ and $y = b''$, in (32), we obtain

$$(c'', b')_3 = (c', b'')_3 = \tfrac{1}{3}\, \Delta_3. \tag{34}$$

The quadri-covariant, Hessian or canonizant, is defined

$$(3_2)\widehat{\ }(x, y)_2 = \begin{vmatrix} 3ax + by\,, & bx + cy \\ bx + cy\,, & cx + 3dy \end{vmatrix}$$

$$= \begin{vmatrix} y^2\,, & xy\,, & x^2 \\ 3a\,, & b\,, & c \\ b\,, & c\,, & 3d \end{vmatrix};$$

or more briefly, $\qquad = (3_2)_2.$* (35)

Expressing this in terms of the auxiliaries we have

$$(3_2)_2 = -\, b'b''x^2 + (b'c' + b''c'')\, xy - c'c''y^2$$

$$= -\,(b'x - c''y)\,(b''x - c'y). \tag{36}$$

The cubicovariant is defined

$$\begin{aligned} (3_3)\widehat{\ }(x, y)_3 = (3_3)_3 = \quad & (2b^3 - 9\,abc + 27\,a^2d)\, x^3 &= \quad & Ax^3 \\ & +\, 3\,(b^2c - 6\,ac^2 + \ \ 9\,abd)\, x^2y & & +\, 3\,Bx^2y \\ & -\, 3\,(bc^2 - 6\,b^2d + \ \ 9\,acd)\, xy^2 & & -\, 3\,Cxy^2 \\ & - \quad (2c^3 - 9\,bcd + 27\,ad^2)\, y^3 & & -\, Dy^3. \end{aligned} \tag{37}$$

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* This is an example of a general notation for covariants and invariants of one or more quantics; thus, $(n_p)\widehat{\ }(x, y)_g$ denotes a covariant of a quantic $(n_1)\widehat{\ }(x, y)_n$ of the nth order, whose weight in coefficients is p, and which is of the gth degree in form. Also $(n_p,\, n'_{p'})\widehat{\ }(x, y)_g$ denotes a covariant of two quantics, one of the nth order entering with weight p and another of the n'th order entering with weight p', the resulting form being of degree g. If the variables need not be shown, the shorter forms $(n_p)_g$, $(n_1)_n$, $(n_p,\, n'_{p'})_g$ may be used, and for invariants $g = 0$. The principal quantic may be denoted $(x, y)_n$, as we have done above.

Now, we have from (19), (20), (21), (24), and (25):

$$b'^3 + b''^3 = 2\,bb'b'' + 3\,a\,(b'c' + b''c'') = 2\,b^3 - 9\,abc + 27\,a^2d = A,$$
$$b'c''^2 + b''^2c' = b\,(b'c' + b''c'') + 6\,ac'c'' = -\,b^2c + 6\,ac^2 - 9\,abd = -\,B,$$
$$b'c''^2 + b''c'^2 = c\,(b'c' + b''c'') + 6\,b'b''d = -\,bc^2 + 6\,b^2d - 9\,acd = -\,C,$$
$$c''^3 + c'^3 = 2\,cc'c'' + 3\,d\,(b'c' + b''c'') = 2\,c^3 - 9\,bcd + 27\,ad^2 = D.$$

Therefore

$$\widehat{(3_3)}(x,\,y)_3 = (3_3)_3 = (b'x - c''y)^3 + (b''x - c'y)^3; \tag{38}$$

and since

$$[(b'x - c''y)^3 + (b''x - c'y)^3]^2 = [(b'x - c''y)^3 - (b''x - c'y)^3]^2 + 4\,(b'x - c''y)^3\,(b''x - c'y)^3,$$

we have Cayley's relation,

$$(3_3)_3^{\,2} = 9\,\Delta_3\,(x,\,y)_3^{\,2} - 4\,(3_2)_2^{\,3}. \tag{39}$$

It is interesting to note the following special values of the cubic and its covariants:

$$\begin{aligned}
&(-\,b,\,3\,a)_3 = A,\\
&(3\,d,\,-\,c)_3 = D,\\
&(c'',\,b')_3 = (c',\,b'')_3 = \tfrac{1}{3}\,\Delta_3,\\
&(c'' - c',\,b' - b'')_3 = 0,
\end{aligned} \tag{40}$$

$$\begin{aligned}
&\widehat{(3_2)}(-\,b,\,3\,a)_2 = -\,b'^2\,b''^2 = -\,A'^2,\\
&\widehat{(3_2)}(3\,d,\,-\,c)_2 = -\,c'^2\,c''^2 = -\,D'^2,\\
&\widehat{(3_2)}(c'',\,b')_2 = \widehat{(3_2)}(c',\,b'')_2 = 0,\\
&\widehat{(3_2)}(c'' - c',\,b' - b'')_2 = -\,\Delta_3,
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\widehat{(3_3)}(-\,b,\,3\,a)_3 = -\,b'^6 - b''^6 = -\,A^2 + 2A'^3,\\
&\widehat{(3_3)}(3\,d,\,-\,c)_3 = c'^6 + c''^6 = D^2 - 2\,D'^3,\\
&\widehat{(3_3)}(c'',\,b')_3 = -\,\Delta_3^{\frac{3}{2}} = -\,\widehat{(3_3)}(c',\,b'')_3,\\
&\widehat{(3_3)}(c'' - c',\,b' - c'')_3 = -\,2\,\Delta_3^{\frac{3}{2}}.
\end{aligned} \tag{42}$$

It is also evident that the linear substitutions to reduce the cubic to the canonical form must be

$$x = c''X + c'\,Y, \tag{43}$$
$$y = b'X + b''\,Y; \tag{44}$$

for we have then

$$\begin{aligned}(x, y)_3 &= (c''X + c'Y, b'X + b''Y)_3 \\ &= (c'', b')_3 X^3 + [c'(3ac''^2 + 2bc''b' + cb'^2) \\ &\qquad + b''(bc''^2 + 2cc''b' + 3db'^2)] X^2 Y \\ &\quad + [c''(3ac'^2 + 2bc'b'' + cb''^2) \\ &\qquad + b'(bc'^2 + 2cc'b'' + 3db''^2)] XY^2 + (c', b'')_3 Y^3 \\ &= \tfrac{1}{3}\Delta_3 (X^3 + Y^3),\end{aligned}$$

by (40), and because each of the middle terms vanishes identically by virtue of (28), (29), (30), and (31). Now, because

$$X = -\frac{b''x - c'y}{b'c' - b''c''} = -\frac{1}{\sqrt{\Delta_3}}(b''x - c'y), \tag{45}$$

$$Y = \frac{b'x - c''y}{b'c' - b''c''} = \frac{1}{\sqrt{\Delta_3}}(b'x - c''y)\,; \tag{46}$$

we have

$$(x, y)_3 = \frac{1}{3\sqrt{\Delta_3}}[(b'x - c''y)^3 - (b''x - c'y)^3]$$

as before.

I shall now discuss some of the principal methods of treating the irreducible case. Of these the most remarkable is, perhaps, that of Guido Weichold, l. c. He employs the same auxiliaries only with different notation, he has

$$b' = -\rho, \quad b'' = -\rho',$$
$$c' = \rho_1, \quad c'' = \rho_1'.$$

He expresses, in terms of the coefficients of the cubic, certain pairs of quantities, such as

$b'b''$ and $b'c'$, or $c'c''$ and $b'c'$, or $b'b''$ and b'^3, or $c'c''$ and c'^3, etc.,

which have a common factor, and one of which, in the irreducible case, is a complex quantity. If at least one root of the cubic is rational, these complex quantities b', b'', c', c'' must be such that, being added to their respective conjugates, an integer results. It must then be possible to determine

$$b' = \overline{b'b'' \,|\, b'c'} = \text{factor common to } b'b'' \text{ and } b'c',$$

or $\qquad c' = \overline{c'c'' \,|\, b'c'} =$ " " " $c'c''$ " $b'c'$, etc.,

by the arithmetical process for finding the greatest common divisor. If this process terminates, the last divisor which divides exactly will be a multiple of the common factor sought, for instance, b'; whence we know also its conjugate b'', and the roots of the cubic, result from (15) or (16). Thus we avoid the determination of these auxiliaries by extraction of a cube root of a complex quantity by (13), (14) or (26), (27), which is obviously impossible. There is, however, this difficulty, that the correct quotient cannot be assumed, because the dividend is given in binomial instead of trinomial form. If, then, the process terminates, it is, in reality, a random success. To make this clear, let

$$b' = \tfrac{1}{2}\,a + \tfrac{1}{2}\sqrt{-3\beta}\,, \qquad b'' = \tfrac{1}{2}\,a - \tfrac{1}{2}\sqrt{-3\beta}\,, \tag{47}$$

$$c' = \tfrac{1}{2}\,\delta \pm \tfrac{1}{2}\sqrt{-3\gamma}\,,^{*} \qquad c'' = \tfrac{1}{2}\,\delta \mp \tfrac{1}{2}\sqrt{-3\gamma}\,; \tag{48}$$

then $$b'b'' = \tfrac{1}{4}\,a^2 + \tfrac{3}{4}\,\beta\,,$$

and $$b'c' = \tfrac{1}{4}\,a\delta \pm \tfrac{1}{4}\,(a\sqrt{\gamma} \mp \delta\sqrt{\beta})\sqrt{-3} \mp \tfrac{3}{4}\sqrt{\beta\gamma}\,.$$

Using $b'b''$ as first divisor we see that the exact quotient of the first division is $a^{-1}\delta$ which will be used if a divides δ exactly; otherwise, the nearest integer will be assumed. But, since the first and third term are merged into one quantity, it is impossible to know either the exact or approximate quotient. This alone does not, however, vitiate the result, provided a correct quotient is used by chance, or otherwise, at some future step. Besides, it is obviously impossible, without some criterion, to know whether the cubic has rational roots or not. In the latter case, Weichold's exceedingly tedious process would never terminate, and this without any indication in the process itself. We need, therefore, a criterion to decide this. The following identity may be used for this purpose:

$$\frac{b'^3 + b''^3}{b' + b''} - b'b'' = (b' + b'')^2 - 4\,b'b'' = (b' - b'')^2.$$

Expressing this by means of (6′), (7′), and (47), we have

$$A a^{-1} - A' = a^2 - 4\,A' = -3\,\beta. \tag{49}$$

Similarly, we have for the reciprocal solution of the cubic

$$D\delta^{-1} - D' = \delta^2 - 4\,D' = -3\,\gamma. \tag{50}$$

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* The sign of $\sqrt{-3\gamma}$ is indefinite unless we form both pairs of auxiliaries; then their connecting condition, as derived by (20), is

$$9\,ad - bc = \tfrac{1}{2}\,(a\delta - 3\sqrt{\beta\gamma}). \tag{20′}$$

This shows a to be an exact divisor of the cubic variant A (also δ of D the cubic retrovariant). If, then, all possible factors of A (or D) are tried by this criterion, and it is not satisfied, we conclude that the cubic cannot have any rational root. It is of course only necessary to try one of these criteria, yet the labor would be very great if each possible factor of A (or D) had actually to be tried, when at most only three can satisfy the criterion. Now it is evident that, if a is a root of the cubic

$$A + 3\,A'a - a^3 = 0\,,$$

then $a \mp n$ will exactly divide the quantity

$$A_{\mp n} = A \pm 3\,A'n \mp n^3\,; \tag{51}$$

so that

$$\frac{A_{\mp n}}{a \mp n} = Q_{\mp n}, \text{ an integer.} \tag{52}$$

Similarly, if

$$D_{\mp n} = D \pm 3\,D'n \mp n^3, \tag{53}$$

then

$$\frac{D_{\mp n}}{\delta \mp n} = R_{\mp n}, \text{ an integer.} \tag{54}$$

Finding, then, also the factors of $A_{-1} = A + 3\,A' - 1$, for example, we can, by comparing them with those of A, exclude a considerable number which it is unnecessary to try by the criterion. If we also find the factors of A_{+1}, and of others of these quantities if necessary, we shall be able to select such factors as satisfy the criterion. There are now three cases:

1) There are no factors of $A_{-n}, \ldots, A_{-1}, A, A_{+1}, \ldots, A_{+n}$ in regular sequence; then we conclude that the cubic can have no rational root.

2) There is but one sequence in these factors; in this case the cubic has one rational root.

3) There are three sequences; which happens if the cubic has three rational roots.

All this is, however, only an application of a well-known method of determining rational roots of any binary quantic, and we see that the knowledge of one root is required, exactly as in Bombelli's method. There is, however, a special advantage in using the criterion in the form given, because it deter-

mines two auxiliaries α, β (or δ, γ) in one operation, whence the roots may be derived by the relations

$$
\begin{aligned}
0 &= 3\,a\,x_1y_1^{-1} + b + \alpha\,, \\
0 &= 3\,a\,x_2y_2^{-1} + b - \tfrac{1}{2}\,\alpha - \tfrac{3}{2}\sqrt{\beta}\,, \\
0 &= 3\,a\,x_3y_3^{-1} + b - \tfrac{1}{2}\,\alpha + \tfrac{3}{2}\sqrt{\beta}\,, \qquad (55)\\
0 &= 3\,d\,y_1x_1^{-1} + c + \delta\,, \\
0 &= 3\,d\,y_2x_2^{-1} + c - \tfrac{1}{2}\,\delta \pm \tfrac{3}{2}\sqrt{\gamma}\,, \\
0 &= 3\,d\,y_3x_3^{-1} + c - \tfrac{1}{2}\,\delta \mp \tfrac{3}{2}\sqrt{\gamma}\,; \qquad (56)
\end{aligned}
$$

or by (33) we have also

$$
\begin{aligned}
x_1y_1^{-1} &= \frac{c'' - c'}{b' - b''} = \mp\sqrt{\frac{\gamma}{\beta}}\,, \\
x_2y_2^{-1} &= \frac{c''\,1^{\frac{1}{3}} - c'\,1^{\frac{2}{3}}}{b'\,1^{\frac{1}{3}} - b''\,1^{\frac{2}{3}}} = \frac{\delta \pm \sqrt{\gamma}}{\alpha - \sqrt{\beta}}\,, \\
x_3y_3^{-1} &= \frac{c''\,1^{\frac{2}{3}} - c'\,1^{\frac{1}{3}}}{b'\,1^{\frac{2}{3}} - b''\,1^{\frac{1}{3}}} = \frac{\delta \mp \sqrt{\gamma}}{\alpha + \sqrt{\beta}}\,; \qquad (57)
\end{aligned}
$$

in which $\sqrt{\beta}$ and $\sqrt{\gamma}$ may be expressed symbolically thus :

$$\sqrt{\beta} = \sqrt{\tfrac{1}{3}\,(A' - A\alpha^{-1} = 4\,A' - \alpha^2)}\,, \qquad (58)$$

$$\sqrt{\gamma} = \sqrt{\tfrac{1}{3}\,(D' - D\delta^{-1} = 4\,D' - \delta^2)}\,. \qquad (59)$$

It is now evident that any examination of a cubic for the purpose of learning the nature of its roots, cannot avoid the actual determination of one of its rational roots, if it has any ; and since after having proved the non-existence of rational roots, Weichold's process cannot terminate and becomes superfluous if there are rational roots, it is in all cases unnecessary to use it ; yet it is one of the most remarkable attempts to solve the irreducible case.

The identity

$$\frac{b'^3 - b''^3}{b' - b''} + b'b'' = 4\,b'b'' + (b' - b'')^2 = (b' + b'')^2\,,$$

or

$$\frac{3\,a\sqrt{\Delta_3}}{\sqrt{-3\,\beta}} + A' = 4\,A' - 3\,\beta = \alpha^2\,, \qquad (60)$$

which shows β to be an exact divisor of the discriminant, might be used, but far less conveniently than (49). This method would be closely related to Kendall's given in the American Journal of Mathematics, Vol. I, p. 285.

The method given by Matthiessen in his great work entitled, Grundzuege der antiken und modernen Algebra, p. 390, is no solution of the irreducible case in the sense in which we have considered it above.

I shall now show the application of our formulæ to the solution of the cubic in its different cases.

1) *No rational root:*

$$0 = x^3 - 5\,xy^2 + 3\,y^3.$$

Here we have $A' = 15$, and $A_{-1} = +125$, with factors 5, 25, 125;

$$A = +\ 81, \quad \text{"} \quad \text{"} \quad 3, 9, 27, 81;$$

$$A_1 = +\ 37, \quad \text{"} \quad \text{"} \quad 37;$$

and since it is impossible to arrange any of these factors into a sequence, there can be no rational root.

2) *One rational root.*

a. Two imaginary roots:

$$0 = x^3 - x^2y - xy^2 - 2\,y^3.$$

Here we have $A' = +4$, $A_{-1} = -54$, with factors 2, 3, 6, 9, 18, 27, 54;

$$A = -65, \quad \text{"} \quad \text{"} \quad 5, 13, 65,$$

$$A_1 = -76, \quad \text{"} \quad \text{"} \quad 2, 4, 19.$$

The only possible sequence is 6, 5, 4; hence $a = -6 + 1 = -5 = -4 - 1$, and to test this by criterion (49), I use the form

$$\begin{array}{rcccl} a = -5 \ | & -65\ | & +13; & a^2 = 25 & \\ & -65 & -\ 4 & -16 & \\ \hline & 0 & +\ 9 & = +\ 9 & = -3\,\beta. \end{array}$$

We have, therefore,

$$b' = -\tfrac{5}{2} + \tfrac{1}{2}\sqrt{9} = -1\,,$$

$$b'' = -\tfrac{5}{2} - \tfrac{1}{2}\sqrt{9} = -4\,;$$

and by (55),

$$0 = 3\,x_1y_1^{-1} - 1 - 1 - 4\,; \qquad \therefore\ x_1y_1^{-1} = 2:$$

$$0 = 3\,x_2y_2^{-1} - 1 + \tfrac{5}{2} - \tfrac{3}{2}\sqrt{-3}\,; \ \therefore\ x_2y_2^{-1} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}:$$

$$0 = 3\,x_3y_3^{-1} - 1 + \tfrac{5}{2} + \tfrac{3}{2}\sqrt{-3}\,; \ \therefore\ x_3y_3^{-1} = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}.$$

Or, we may solve reciprocally, and have

$$D' = -5\,,\ D_{-1} = +108, \text{ with factors } 2, 3, 4, 9, 18, 27, 54\,;$$
$$D = +124, \quad \text{"} \quad \text{"} \quad 2, 4, 31, 62\,;$$
$$D_1 = +140, \quad \text{"} \quad \text{"} \quad 2, 4, 5, 7, 10, 14, 20, 28, 35, 70.$$

Here the only sequence is 3, 4, 5 ; hence $\delta = 3 + 1 = 4 = 5 - 1$; and testing this by (50), we have

$$\begin{array}{rrrrl} \delta = +4 \,| & +124 \,| & +31\,; & \delta^2 = & 16 \\ & +124 & +5 & & +20 \\ \hline & 0 & +36 & = & +36 = -3\gamma. \end{array}$$

We have, therefore,

$$c' = 2 - \sqrt{9} = -1\,,$$
$$c'' = 2 + \sqrt{9} = +5\,;$$

and by (56),

$$0 = -6\,y_1x_1^{-1} - 1 + 4\,; \qquad \therefore\ y_1x_1^{-1} = \tfrac{1}{2}\,;$$
$$0 = -6\,y_2x_2^{-1} - 1 - 2 - 3\sqrt{-3}\,;\ \therefore\ y_2x_2^{-1} = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}\,;$$
$$0 = -6\,y_3x_3^{-1} - 1 - 2 + 3\sqrt{-3}\,;\ \therefore\ y_3x_3^{-1} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}\,;$$

or combining both solutions, we have, by (57),

$$x_1y_1^{-1} = \pm\sqrt{\frac{36}{9}} = \pm 2\,.$$

But the sign criterion (20′) gives

$$\sqrt{\beta\gamma} = \tfrac{2}{3}(1 - 10 + 18) = +6\,;\ \therefore\ \sqrt{\beta} = +\sqrt{-3}, \text{ and } \sqrt{\gamma} = -2\sqrt{-3}\,;$$

hence $$x_1y_1^{-1} = +2\,;$$

also, $$x_2y_2^{-1} = \frac{4 - 2\sqrt{-3}}{-5 - \sqrt{-3}} = \frac{-14 + 14\sqrt{-3}}{28} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}\,,$$

$$x_3y_3^{-1} = \frac{4 + 2\sqrt{-3}}{-5 + \sqrt{-3}} = \frac{-14 - 14\sqrt{-3}}{28} = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}\,.$$

We may, however, employ resolvent (22) with advantage. We have

$$0 = p^2 + 19p - 20\,;$$

hence $\left.\begin{matrix} p' \\ p'' \end{matrix}\right\} = -\tfrac{19}{2} \pm \tfrac{1}{2}\sqrt{361+80} = -\tfrac{19}{2} \pm \tfrac{12}{2} = \left\{\begin{matrix} +\ \ 1\,, \\ -20\,; \end{matrix}\right.$

and by (26) $b' = \sqrt[3]{-4+3} = -1\,,$

$$b'' = \sqrt[3]{-4-60} = -4\,;$$

or by (27) $c' = \sqrt[3]{5-6} = -1\,,$

$$c'' = \sqrt[3]{5+120} = +5\,.$$

We obtain then the roots from either pair of auxiliaries by (15) or (16). Using, however (33), we have

$$x_1 y_1^{-1} = \frac{+5+1}{-1+4} = +2\,,$$

$$x_2 y_2^{-1} = \frac{-\tfrac{5}{2}+\tfrac{5}{2}\sqrt{-3}-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}}{+\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}-\tfrac{4}{2}-\tfrac{4}{2}\sqrt{-3}} = \frac{-6+4\sqrt{-3}}{-3-5\sqrt{-3}} = \frac{-42-42\sqrt{-3}}{84}$$
$$= -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}\,,$$

and

$$x_3 y_3^{-1} = -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3}\,.$$

b. Two real irrational roots:

$$0 = 12x^3 - 4x^2y - 14xy^2 - 4y^3\,.$$

Here we have $A' = 520$;

$A_{-1} = -20169$, with factors 3, 9, 27, 81, 83, 243, etc.

$A = -21728$, " " 2, 4, 7, 8, 14, 16, 28, 32, 56, etc.

$A_1 = -23287$, " " 29, 803, etc.

The only sequence is 27, 28, 29; hence $a = 27 + 1 = 28 = 29 - 1$, and by (50), we have

$$\begin{array}{r|r|rrr}
a = +28 & -21728 & -776 & a^2 = & 784 \\
 & -196 & -520 & & -2080 \\ \hline
 & 212 & -1296 & = & -1296 = -3\beta. \\
 & 196 & & & \\ \hline
 & 168 & & & \\
 & 168 & & & \\ \hline
 & 0 & & &
\end{array}$$

We have, therefore,

$$b' = 14 + 18\sqrt{-1}\,,$$

$$b'' = 14 - 18\sqrt{-1}\,;$$

and by (55) we have

$$0 = 36\,x_1y_1^{-1} - 4 + 28\,; \qquad \therefore\ x_1y_1^{-1} = -\tfrac{2}{3}\,;$$

$$0 = 36\,x_2y_2^{-1} - 4 - 14 - 18\sqrt{3}\,;\ \therefore\ x_2y_2^{-1} = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,;$$

$$0 = 36\,x_3y_3^{-1} - 4 - 14 + 18\sqrt{3}\,;\ \therefore\ x_3y_3^{-1} = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}\,.$$

For the reciprocal solution we have $D' = 148$, $D = 1712$, and find in the same manner $\delta = -4$, $-3\gamma = -576$; and if we wish to satisfy condition (20) we must put

$$c' = -2 + 12\sqrt{-1}\,,$$

$$c'' = -2 - 12\sqrt{-1}\,.$$

To show, also, how Weichold's method is applied, we have

$$9\,ad - bc = -488\,;$$

whence by (23) $\qquad b'c' = -244 + 132\sqrt{-1}\,,$

and, also, $\qquad c'c'' = D' = 148\,.$

We shall have, then, symbolically

$$\overline{-244 + 132\sqrt{-1}\mid 148} = kc'\,;$$

whence, using 148 for first divisor,

$$\frac{-244 + 132\sqrt{-1}}{148} = -2 + \sqrt{-1} + \frac{52 - 16\sqrt{-1}}{148}\,;$$

$\therefore\ 52 - 16\sqrt{-1} = 4\,(13 - 4\sqrt{-1}) =$ first remainder.

Dividing this, as simplified, into former divisor, we have

$$\frac{148}{13 - 4\sqrt{-1}} = \frac{52 + 16\sqrt{-1}}{5} = 10 + 3\sqrt{-1} + \frac{2 + \sqrt{-1}}{5}$$

$$= 10 + 3\sqrt{-1} + \frac{6 + \sqrt{-1}}{13 - 4\sqrt{-1}}\,;$$

$\therefore\ 6 + \sqrt{-1} =$ second remainder, and dividing this into former divisor, we have

$$\frac{13 - 4\sqrt{-1}}{6 + \sqrt{-1}} = 2 - \sqrt{-1}\,.$$

Here the process terminates, therefore $6+\sqrt{-1}=kc'$; and since by (25)

$$c'^3 = cD' + 3\,b'c'd = -2072 + 2928 - 1584\sqrt{-1}$$
$$= 856 - 1584\sqrt{-1}$$
$$= k^3(198 - 107\sqrt{-1});$$

$$\therefore\ k = 2\sqrt{-1} \text{ and } c' = -2 + 12\sqrt{-1}, \text{ as before.}$$

(3). *Three rational roots*

$$0 = x^3 - 3\,x^2y - 60\,xy^2 - 100\,.$$

We have $A' = 189$, $A_{-1} = -3808 = 2.4.7.8.14.16.17.28.32.34\ldots,$

$$A = -4374 = 2.3.6.9.18.27.54.81.162\ldots,$$
$$A_1 = -4940 = 2.4.5.10.13.19.20.26.38.52\ldots,$$
$$A_2 = -5500 = 2.4.5.10.11.20.22.25.40.44\ldots,$$
$$A_3 = -6048 = 2.3.4.6.7.9.12.14.21.24.27\,.$$

The only sequences that can be formed are (7, 6, 5, 4, 3), (8, 9, 10, 11, 12), (17, 18, 19, 20, 21), (28, 27, 26, 25, 24), therefore we may try $a = -6$, or $= +9$, or $= +18$, or $= -27$. Trying -6 we have

$$\begin{array}{r|r|rr} a = -6 & -4374 & +729 & a^2 = \quad 36 \\ & -4374 & -189 & -756 \\ \hline & 0 & +540 & \text{not} = -720 \end{array}$$

$\therefore\ a = -6$ does not satisfy.

$$\begin{array}{r|r|rr} a = +9 & -4374 & -486 & a^2 = \quad 81 \\ & -4374 & -189 & -756 \\ \hline & 0 & -675 & = -675 \end{array}$$

$\therefore\ a = +9$, and $0 = 3\,x_1y_1^{-1} - 3 + 9$; therefore, $x_1y_1^{-1} = -2$.

$$\begin{array}{r|r|rr} a = +18 & -4374 & -243 & a^2 = \quad 324 \\ & -4374 & -189 & -756 \\ \hline & 0 & -432 & = -432 \end{array}$$

$\therefore\ a = +18$, and $0 = 3\,x_2y_2^{-1} - 3 + 18$; therefore, $x_2y_2^{-1} = -5$.

$$\begin{array}{r|r|rr} a = -27 & -4374 & +162 & a^2 = \quad 729 \\ & -4374 & -189 & -756 \\ \hline & 0 & -\ \ 27 & = -\ \ 27 \end{array}$$

$\therefore\ a = -27$, and $0 = 3\,x_3y_3^{-1} - 3 - 27$; therefore, $x_3y_3^{-1} = +10$.

This example is remarkable for having a persistent false sequence, for we have also 2 a factor of $A_4 = -6578$, and 1 a factor of $A_5 = -7084$. But then 0 cannot be a factor of A_6, which breaks up the sequence. Since, also, 8 is a factor of $A_{-2} = -3248$, 9 a factor of $A_{-3} = -2700$, 10 a factor of $A_{-4} = -2170$, but 11 no factor of $A_{-5} = -1664$, it persists through ten steps.*

The trigonometric method of solving a cubic by the trisection of an angle is usually considered a solution of the irreducible case. It is as proper and as convenient as any known method in the case of three irrational roots, but does not give the true form of the roots in other cases ; yet the relations of the roots to these angles are so remarkable, that some space may be devoted to showing them. We have by analytical trigonometry,

$$0 = 4 \cos^3 \tfrac{1}{3} \varphi - 3 \cos \tfrac{1}{3} \varphi - \cos \varphi. \tag{61}$$

Comparing this with (31) we have

$$0 = 3\, a\, xy^{-1} + b + 2 \sqrt{A'} \cos \tfrac{1}{3} \varphi\,, \tag{62}$$

$$0 = A - 2A'^{\frac{3}{2}} \cos \varphi\,; \tag{63}$$

and therefore

$$b' = \sqrt{A'}\, e^{\frac{1}{3}\phi i},$$

$$b'' = \sqrt{A'}\, e^{-\frac{1}{3}\phi i}. \tag{64}$$

Similarly, by comparing with (5′) we may assume

$$0 = 2 \sqrt{D'} \cos \tfrac{1}{3} \psi + c + 3\, d\, yx^{-1}, \tag{65}$$

$$0 = D - 2\, D'^{\frac{3}{2}} \cos \psi\,; \tag{66}$$

and therefore

$$c' = \sqrt{D'}\, e^{\frac{1}{3}\psi i},$$

$$c'' = \sqrt{D'}\, e^{-\frac{1}{3}\psi i}. \tag{67}$$

By (19), (20), (21) there must be the following relations between these angles:

$$\begin{aligned} b \sqrt{A'} \cos \tfrac{1}{3} \varphi &= A' \cos \tfrac{2}{3} \varphi - 3\, a \sqrt{D'} \cos \tfrac{1}{3} \psi\,, \\ - b \sqrt{A'} \sin \tfrac{1}{3} \varphi &= A' \sin \tfrac{2}{3} \varphi - 3\, a \sqrt{D'} \sin \tfrac{1}{3} \psi\,, \end{aligned} \tag{19'}$$

$$\begin{aligned} & bc + 2 \sqrt{A'D'} \cos \tfrac{1}{3} (\varphi + \psi) = 9\, ad\,, \\ & b \sqrt{D'} \cos \tfrac{1}{3} \psi + \sqrt{A'D'} \cos \tfrac{1}{3} (\varphi - \psi) + c \sqrt{A'} \cos \tfrac{1}{3} \varphi = 0\,, \\ & b \sqrt{D'} \sin \tfrac{1}{3} \psi + \sqrt{A'D'} \sin \tfrac{1}{3} (\varphi - \psi) - c \sqrt{A'} \sin \tfrac{1}{3} \varphi = 0\,, \end{aligned} \tag{20'}$$

$$\begin{aligned} c \sqrt{D'} \cos \tfrac{1}{3} \psi &= D' \cos \tfrac{2}{3} \psi - 3\, d \sqrt{A'} \cos \tfrac{1}{3} \varphi\,, \\ - c \sqrt{D'} \sin \tfrac{1}{3} \psi &= D' \sin \tfrac{2}{3} \psi - 3\, d \sqrt{A'} \sin \tfrac{1}{3} \varphi\,. \end{aligned} \tag{21'}$$

* We might, however, have saved ourselves the computation and factoring of the A's external to A_{-1}, A, A_1 by using the condition $a_1 a_2 a_3 = A$, which is only satisfied by 9, 18, 27.

Also, (24) and (25) become

$$bA' = A'^{\frac{3}{2}} e^{\phi i} - 3\, a \sqrt{A'D'}\, e^{\frac{1}{3}(\phi+\psi)i} = A'^{\frac{3}{2}} e^{-\phi i} - 3\, a \sqrt{A'D'}\, e^{-\frac{1}{3}(\phi+\psi)i},$$

$$cD' = D'^{\frac{3}{2}} e^{\psi i} - 3\, d \sqrt{A'D'}\, e^{\frac{1}{3}(\phi+\psi)i} = D'^{\frac{3}{2}} e^{-\psi i} - 3\, d \sqrt{A'D'}\, e^{-\frac{1}{3}(\phi+\psi)i};$$

or

$$bA' = A'^{\frac{3}{2}} \cos\varphi - 3\, a \sqrt{A'D'} \cos \tfrac{1}{3}(\varphi + \psi),$$

$$0 = A'^{\frac{3}{2}} \sin\varphi - 3\, a \sqrt{A'D'} \sin \tfrac{1}{3}(\varphi + \psi), \tag{24'}$$

$$cD' = D'^{\frac{3}{2}} \cos\psi - 3\, d \sqrt{A'D'} \cos \tfrac{1}{3}(\varphi + \psi),$$

$$0 = D'^{\frac{3}{2}} \sin\psi - 3\, d \sqrt{A'D'} \sin \tfrac{1}{3}(\varphi + \psi). \tag{25'}$$

These relations suggest many ways of solving the cubic, among which the following four are perhaps the most convenient and elegant :—

(1) Compute

$$\cos\varphi = \frac{A}{2\, A'^{\frac{3}{2}}}; \tag{68}$$

then we have

$$x_1 y_1^{-1} = -\tfrac{1}{3}\, a^{-1} (b + 2\sqrt{A'} \cos \tfrac{1}{3}\varphi),$$

$$x_2 y_2^{-1} = -\tfrac{1}{3}\, a^{-1} (b + 2\sqrt{A'} \cos \tfrac{1}{3}(\varphi + 2\pi)),$$

$$x_3 y_3^{-1} = -\tfrac{1}{3}\, a^{-1} (b + 2\sqrt{A'} \cos \tfrac{1}{3}(\varphi - 2\pi)). \tag{69}$$

(2) Compute

$$\cos\psi = \frac{D}{2\, D'^{\frac{3}{2}}}; \tag{70}$$

then we have

$$y_1 x_1^{-1} = -\tfrac{1}{3}\, d^{-1} (2\sqrt{D'} \cos \tfrac{1}{3}\psi + c),$$

$$y_2 x_2^{-1} = -\tfrac{1}{3}\, d^{-1} (2\sqrt{D'} \cos \tfrac{1}{3}(\psi + 2\pi) + c),$$

$$y_3 x_3^{-1} = -\tfrac{1}{3}\, d^{-1} (2\sqrt{D'} \cos \tfrac{1}{3}(\psi - 2\pi) + c). \tag{71}$$

(3) Compute

$$\cos \tfrac{1}{3}(\varphi + \psi) = \frac{9\, ad - bc}{2\sqrt{A'D'}}, \tag{72}$$

and

$$\sin\varphi = \frac{3\, a \sqrt{D'}}{A'} \sin \tfrac{1}{3}(\varphi + \psi); \tag{73}$$

then the roots are found by (69):

or compute

$$\sin\psi = \frac{3\, d \sqrt{A'}}{D'} \sin \tfrac{1}{3}(\varphi + \psi); \tag{74}$$

then the roots are found by (71).

(4) Compute $\frac{1}{3}(\varphi + \psi)$ by (72), φ by (68) or (73), and ψ by (70) or (74). Assume ψ, so that

$$\tfrac{1}{3}\varphi + \tfrac{1}{3}\psi = \tfrac{1}{3}(\varphi + \psi)\,;$$

then we have

$$x_1 y_1^{-1} = \frac{c'' - c'}{b' - b''} = -\,\frac{\sqrt{D'}\sin\frac{1}{3}\psi}{\sqrt{A'}\sin\frac{1}{3}\varphi},$$

$$x_2 y_2^{-1} = \frac{c''\,1^{\frac{1}{3}} - c'\,1^{\frac{2}{3}}}{b'\,1^{\frac{1}{3}} - b''\,1^{\frac{2}{3}}} = -\,\frac{\sqrt{D'}\sin\frac{1}{3}(\psi - 2\pi)}{\sqrt{A'}\sin\frac{1}{3}(\varphi + 2\pi)},$$

$$x_3 y_3^{-1} = \frac{c''\,1^{\frac{2}{3}} - c'\,1^{\frac{1}{3}}}{b'\,1^{\frac{2}{3}} - b''\,1^{\frac{1}{3}}} = -\,\frac{\sqrt{D'}\sin\frac{1}{3}(\psi + 2\pi)}{\sqrt{A'}\sin\frac{1}{3}(\varphi - 2\pi)}. \tag{75}$$

For illustration of this last elegant method I shall solve the cubic already solved above

$$0 = 12\,x^3 - 4\,x^2y - 14\,xy^2 - 4\,y^3.$$

We have here $A' = 520$, $A = -21728$, $9\,ad - bc = -488$, and $D' = 148$, $D = 1712$; and the remaining part of the solution is done by logarithms in the following form :—

$$\begin{array}{rl|rl|rl}
\log\frac{1}{2}A = & 4.03599_n & \log\frac{1}{2}(9\,ad - bc) = & 2.38739_n & \log\frac{1}{2}D = & 2.93247 \\
-\frac{3}{2}\log A' = & -4.07400 & -\frac{1}{2}\log A'D' = & -2.44313 & -\frac{3}{2}\log D' = & -3.25539 \\
\hline
\log\cos\varphi = & 9.96199_n & \log\cos\frac{1}{3}(\varphi + \psi) = & 9.94426_n & \log\cos\psi = & 9.67708 \\
\frac{1}{3}\varphi = & 187^\circ\ 52' & \frac{1}{3}(\varphi + \psi) = & 208^\circ\ 25' & \frac{1}{3}\psi = & 20^\circ\ 32'
\end{array}$$

$$\begin{array}{rl|rl}
\log\sin\frac{1}{3}\varphi = & 9.13630_n & \log\sin\frac{1}{3}\psi = & 9.54500 \\
\log\sin\frac{1}{3}(\varphi + 2\pi) = & 9.89732_n & \log\sin\frac{1}{3}(\psi - 2\pi) = & 9.99404_n \\
\log\sin\frac{1}{3}(\varphi - 2\pi) = & 9.96676 & \log\sin\frac{1}{3}(\psi + 2\pi) = & 9.80320 \\
\frac{1}{2}\log A' = & 1.35800 & \frac{1}{2}\log D' = & 1.08513 \\
\hline
\log\sqrt{A'}\sin\frac{1}{3}\varphi = & 0.49430_n & \log\sqrt{D'}\sin\frac{1}{3}\psi = & 0.63013 \\
\log\sqrt{A'}\sin\frac{1}{3}(\varphi + 2\pi) = & 1.25532_n & \log\sqrt{D'}\sin\frac{1}{3}(\psi - 2\pi) = & 1.07917_n \\
\log\sqrt{A'}\sin\frac{1}{3}(\varphi - 2\pi) = & 1.32476 & \log\sqrt{D'}\sin\frac{1}{3}(\psi + 2\pi) = & 0.88833 \\
\log(-x_1y_1^{-1}) = & 0.13583_n & x_1y_1^{-1} = & +1.3672 \\
\log(-x_2y_2^{-1}) = & 9.82385 & x_2y_2^{-1} = & -0.6666 \\
\log(-x_3y_3^{-1}) = & 9.56357 & x_3y_3^{-1} = & -0.3661
\end{array}$$

Comparing these with the values above, we notice that they are here obtained in a different order.